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(2+1)-Gravity with moving particles in an instantaneous gauge *

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Abstract

By defining a regular gauge which is conformal-like and provides instantaneous field propagation, we investigate classical solutions of (2+1)-Gravity coupled to arbitrarily moving point-like particles. We show how to separate field equations from self-consistent motion and we provide a solution for the metric and the motion in the two-body case with arbitrary speed, up to second order in the mass parameters.

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In the past few years, much attention has been devoted to the gravitational problem in 2+1-dimensions [1]-[10], mostly because it shows a few simplifying features which may allow a treatment of the quantum problem.

First of all, the three-dimensional geometry is characterized by the fact that the space is flat outside the matter sources [1]-[3] . This implies that the dynamics of pointlike particles can be made locally trivial and should be determined at large by the global structure of space-time, as suggested by its connection with the (topological) Chern-Simons Poincaré gauge theory [4]-[5].

Secondly, the perturbative quantum problem [5]-[8] is characterized by the absence of (transverse) gravitons. This lack of graviton radiation makes the infrared properties of the theory much "potential" like and may allow a quantum treatment with naive definition of matter asymptotic states.

The first feature has been used to construct general classical solutions for N moving (massive) particles [9]-[10]. For a single particle, one has a cut Minkowskian space-time where the two edges of the cut are related by a rotation in the static case, corresponding to the deficit angle of a conical space and, more generally, by a Lorentz transformation for non-vanishing speed.

Many particles solution can thus be obtained by superimposing in a linear way the various cuts or tails attached to the particle trajectories. This simple linear description is obtained at the expense of singularities and/or multivaluedness of the connection matrix (and of the metric tensor) along the above-mentioned cuts or tails, even for the case of massive particles, where such singularities are not possibly induced by the $v \to c$ limit.

In other words, this class of exact solutions is obtained by choosing a singular gauge in which the metric and the connection are singular even outside the particle trajectories.

By contrast, N-particle classical solutions are fully non-linear and non trivial in regular gauges, in which the particle sites are the only isolated singularities. A method for constructing the non linear coordinate transformation from singular to regular gauges was given in [9], where also the explicit solution was exhibited in the massless limit (see also [11])

The purpose of the present paper is to investigate classical metric and motion in a regular gauge which reduces to the conformal one in the static limit, and is similar to the one used by two of us [12] to discuss the quasi-static case to all orders.

The characteristic feature of this gauge is that it yields an instantaneous propagation for arbitrarily moving particles, and also has a diagonal space part, thus generalizing the conformal gauge. As a consequence, we are able to split the Einstein equations in a set of

four, which determines the metric, and in a remaining set, which determines the motion.

Needless to say, the considerable simplification mentioned above is due to the 3-dimensional nature of the problem, and in particular to the fact that, for a given wave vector there is only one transverse direction, which is unable to propagate physical tensor waves. Nevertheless, it is interesting that this procedure allows to find - at least perturbatively - a regular metric, and to set up the equations of motion in a Newtonian way, for an arbitrary set of moving particles.

In this paper we limit ourselves to the two-body problem with masses m_1, m_2 and arbitrary speed, and we solve for the metric and for the motion up to second order in the mass parameters Gm_i. We also provide the corresponding expression for the scattering angle.

The contents of the paper are as follows. In Sec. II we define our gauge choice, and we describe the corresponding field equations with instantaneous propagation and the particle's equations of motion. In Sec. III we set up the perturbative treatment of our problem and we derive the first order results for both metric and motion. In Sec. IV we derive our main results for the metric tensor and the connection for arbitrary speed, and up to second order in Gm_i. Finally, in Sec. V we discuss the equations of motion and the ensuing scattering angle up to second order and we outline possible developments. Some details of the calculations are deferred to Appendices A and B.

1 An instantaneous gauge for moving particles: general features

For the purpose of orientation, let us recall the static many-particle solutions in (2+1)-dimensions. They were first found [1]-[2] in the conformal gauge, defined by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\phi}\delta_{ij} \end{pmatrix} (\mu, \nu = 0, 1, 2; i, j = 1, 2). \tag{1.1}$$

In this gauge, for one particle at rest in the origin one simply finds

$$e^{2\phi} = \alpha^2 R^{-8Gm}, \qquad \alpha = 1 - 4Gm, \tag{1.2}$$

where $R^2 = \mathbf{x}^2$, so that

$$ds^2 = dt^2 - \alpha^2 R^{-8Gm} d\mathbf{x}^2. \tag{1.3}$$

This proper-time interval can be related, by a redefinition of the radial coordinate

$$r = R^\alpha \tag{1.4}$$

to the conical gauge expression

$$ds^2 = dt^2 - dr^2 - \alpha^2 r^2 d\theta^2, \tag{1.5}$$

thus yielding the customary description of space-time characterized by the deficit angle

$$2\pi - 2\pi\alpha = 8\pi G m. \tag{1.6}$$

For many particles at rest at $\mathbf{x} = \xi_i$, the conformal factor is multiplicative, or equivalently ϕ is additive, i.e.,

$$\phi = -4G \sum_i m_i \log |\mathbf{x} - \xi_i| + \text{ const.}. \tag{1.7}$$

The corresponding conical description was given in [9] by the Λ-mapping method, and involves a slightly more complicated coordinate transformation.

Here, we are interested in generalizing the conformal gauge to allow a reasonably simple description of moving particles. In this general case, we can always reduce the spatial part of the metric to diagonal form, or, by using complex $z, \bar{z}$ coordinates, we can set

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \tag{1.8}$$

and we still have the freedom of an additional gauge condition. However, in general the mixed space-time components will be non-vanishing, and we parametrize

$$g_{00} = \alpha^2 - e^{2\phi}\beta\bar{\beta}, \quad g_{0z} = \frac{1}{2}\bar{\beta}e^{2\phi}, \quad g_{0\bar{z}} = \frac{1}{2}\beta e^{2\phi}, \quad g_{z\bar{z}} = -\frac{1}{2}e^{2\phi}, \tag{1.9}$$

where $\alpha(z, \bar{z}, t)$ and $\phi(z, \bar{z}, t)$ are real functions and $\beta(z, \bar{z}, t)$ is complex. In this notation, the full determinant and the one for the spatial part are given by

$$| \, g \, | = \frac{1}{4} \; \alpha^2 \; e^{4\phi}, \quad | \, g_{ij} \, | = \frac{1}{4} \; e^{4\phi}, \tag{1.10}$$

and the line element takes the form

$$ds^2 = \alpha^2 dt^2 - e^{2\phi} | \, dz - \beta dt \, |^2. \tag{1.11}$$

The remaining gauge condition will be chosen so as to yield instantaneous propagation in the equations of motion. This is possible in (2+1) dimensions because there are not enough transverse coordinates to allow the propagation of tensor waves, for which a retarded propagator would be needed.

In order to understand better this point it is convenient to rewrite the Einstein-Hilbert action by splitting the scalar curvature $R^{(3)}$ into its space part $R^{(2)}$ and a mixed space-time part as follow [13] ($8\pi G = 1$)

$$S \;=\; -\frac{1}{2}\int\sqrt{|\,g\,|}R^{(3)} \;=\; -\frac{1}{2}\int\sqrt{|\,g\,|}\left[R^{(2)}+\left((TrK)^2-TrK^2\right)\right]d^3x, \qquad (1.12)$$

where we have dropped a total derivative [13] giving rise to a boundary term. In Eq. (1.12) we have introduced the extrinsic curvature tensor by the expression

$$K_{ij}=\sqrt{\frac{|g_{ij}|}{g}}\frac{1}{2}\left(\nabla_i^{(2)}g_{0j}+\nabla_j^{(2)}g_{0i}-\partial_0 g_{ij}\right), \qquad (i,j=1,2), \qquad (1.13)$$

where we denote by $\nabla_i^{(2)}$ covariant derivatives in the space part of the metric which is also used to raise and lower the space indices i.

By using the fact that the only nonvanishing component of the 2-dimensional connection in the gauge (1.8) is $\Gamma^z_{zz}=\partial_z\log g_{z\bar{z}}$ and its complex coniugate, it is easy to realize that the matrix (1.13) takes the simple form

$$\begin{aligned} K_{zz} &= \frac{1}{2\alpha}e^{2\phi}\partial_z\bar{\beta}, \qquad K_{\bar{z}\bar{z}}=\bar{K}_{zz}, \\ K_{z\bar{z}} &\equiv K(z,\bar{z},t)=\frac{1}{2\alpha}\left(\partial_z g_{0\bar{z}}+\partial_{\bar{z}}g_{0z}-\partial_0 g_{z\bar{z}}\right)=\frac{1}{\alpha}\Gamma_{0,z\bar{z}}. \end{aligned} \qquad (1.14)$$

Therefore , in this 3-dimensional case, time derivatives only occur in (1.12) through the expression of K in (1.14). We shall thus set $K=0$, i.e.

$$\partial_{\bar{z}}(\bar{\beta}e^{2\phi})+\partial_z(\beta e^{2\phi})+\partial_0(e^{2\phi})=0 \qquad (1.15)$$

as additional gauge condition.

By using (1.8) and (1.15), the action (1.12) becomes simply

$$S=\int d^3x\left[-\alpha\nabla^2\phi+\frac{e^{2\phi}}{\alpha}|\partial_z\bar{\beta}|^2\right]. \qquad (1.16)$$

Since the form (1.15) of the action does not contain time derivatives, it is now obvious that the propagation of the fields α,β,ϕ is instantaneous. As a matter of fact, by adding point-like matter sources, the Einstein equations derived from (1.16) are

$$\begin{aligned} \nabla^2\phi &+ \alpha^{-2}e^{2\phi}\partial_z\bar{\beta}\partial_{\bar{z}}\beta=-|g|e^{-2\phi}T^{00}, \\ \nabla^2\beta &+ 4(2\partial_z\phi-\frac{1}{\alpha}\partial_z\alpha)\partial_{\bar{z}}\beta=-2|g|e^{-2\phi}(T^{0z}-\beta T^{00}), \\ \nabla^2\alpha &- \frac{2e^{2\phi}}{\alpha}\partial_z\bar{\beta}\partial_{\bar{z}}\beta=\alpha^{-1}|g|(T^{z\bar{z}}-\beta T^{0\bar{z}}-\bar{\beta}T^{0z}+\beta\bar{\beta}T^{00}), \end{aligned} \qquad (1.17)$$

where $\nabla^2 \equiv 4\partial_z \partial_{\bar{z}}$ denotes the Laplacian,

$$T^{\mu\nu} = \frac{1}{\sqrt{|g|}} \sum_{(i)} m_i \left(\frac{dt}{ds_i}\right) \dot{\xi}_i^\mu \dot{\xi}_i^\nu \delta^2(\mathbf{x} - \xi_\mathbf{i}(t_i)), \quad (i = 1, ..., N) \tag{1.18}$$

is the energy-momentum tensor, $\mathbf{x} = \xi_\mathbf{i}(t), v_i, s_i$ are the particle trajectories, velocities and proper time, and $\dot{\xi}_i^\mu = (1, v_i)$. It is apparent from (1.17) that the fields α, β, ϕ can now be derived as functions of the trajectories $\xi_i(t)$ and velocities $v_i(t)$ for any given time. It remains to be checked, however, that the gauge conditions are consistent with the equations of motion, and that the energy momentum tensor is conserved.

Let us first remark that, by setting $g_{zz} = g_{\bar{z}\bar{z}} = 0$ (Eq. (1.8)) we have lost the Einstein equation for the corresponding components of the Ricci tensor $R_{\mu\nu}$, which should therefore be added as constraints, i.e.,

$$R_{zz} = T_{zz}, \qquad R_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}. \tag{1.19}$$

Furthermore, since the action (1.12) is in general quadratic in the quantity K given by Eq.(1.14), the additional condition $K = 0$ of Eq. (1.15) is consistent automatically with the full equations of motion.

It is now not difficult to check (Appendix A) that the constraints (1.19) and the condition (1.15) are enough to provide the t-dependence of the trajectories, and with proper asymptotic conditions, are indeed equivalent to the covariant conservation of the energy-momentum tensor, which in turn implies the geodesic equations

$$\frac{d^2\xi_i^\mu}{d{s_i}^2} + (\Gamma^\mu_{\alpha\beta})_i \frac{d\xi_i^\alpha}{ds_i} \frac{d\xi_i^\beta}{ds_i} = 0, \qquad (i = 1, ..., N) \tag{1.20}$$

in the fields provided by Eq. (1.17).

Therefore our procedure will be to determine first the four fields $\alpha, \beta, \bar{\beta}, \phi$ from Eq.(1.17) in terms of the trajectories at a given time, and then to determine the trajectories themselves from the geodesic equations in the self-consistent fields.

This separation of the field equations (1.17) from the equations of motion (1.20) is essentially due to the 3-dimensional nature of the problem, and is the key advantage of the conformal-like gauge that we are using. Note that, in principle, this method allows to find a regular metric and the corresponding motion for a general set of N moving particles. However, we shall focus in the following on the perturbative expansion for the two-body system.

2 Lowest order metric and two-particle motion

The perturbative expansion in Gm_i of Eqs. (1.17) and (1.20) is set up iteratively around Minkowskian metric and linear motion and is rather straightforward. In fact, by using the espression (1.11) of the proper time, we obtain

$$\frac{dt}{ds_i} = \left(\alpha^2 - e^{2\phi}|v_i - \beta|^2\right)^{-\frac{1}{2}}|_i, \tag{2.1}$$

and hence the coefficients of the source terms in the r.h.s. of (1.17) can all be expressed in terms of ξ_i, v_i and of the fields themselves, evaluated at ξ_i. As a consequence the (n)-th iteration determines the source for the (n+1)-th, always through equations of Poisson type.

At first order in Gm_i, we can use the Minkowskian form of proper time

$$\frac{dt}{ds_i} = \gamma_i = (1 - v_i^2)^{-\frac{1}{2}} \tag{2.2}$$

to rewrite Eq. (1.17) in the linearized form,

$$\begin{aligned}
\nabla^2\phi^{(1)} &= -\sum_{i=1}^{2} \gamma_i m_i \delta^{(2)}(\mathbf{x} - \xi_\mathbf{i}), \\
\nabla^2\beta^{(1)} &= -2\sum_{i=1}^{2} \gamma_i m_i v_i \delta^{(2)}(\mathbf{x} - \xi_\mathbf{i}), \\
\nabla^2\alpha^{(1)} &= \sum_{i=1}^{2} \gamma_i m_i v_i \bar{v}_i \delta^{(2)}(\mathbf{x} - \xi_\mathbf{i}).
\end{aligned} \tag{2.3}$$

Here the inversion of the Laplacian is essentially unique (see later), and the metric can be solved in terms of the basic fields

$$\phi_i = -4Gm_i \log|\mathbf{x} - \xi_\mathbf{i}| \tag{2.4}$$

as follows

$$\begin{aligned}
\phi^{(1)} &= \sum_{i=1}^{2} \gamma_i \phi_i, \\
\beta^{(1)} &= \sum_{i=1}^{2} 2\gamma_i v_i \phi_i,
\end{aligned}$$

$$\alpha^{(1)} = -\sum_{i=1}^{2} \gamma_i v_i \bar{v}_i \phi_i. \tag{2.5}$$

It is a matter of inspection to realize that these solutions solve the constraints (1.19) and the gauge condition (1.15) identically.

The expression (2.5) is unique, up to the addition of harmonic solutions of the homogeneous equations in (2.3) which can be reduced to time-dependent constants by requiring that the spatial connections $\Gamma^z_{\alpha\beta}$ vanish at space infinity, or in other terms that rotations are absent at large distances. In principle, this asymptotic condition still leaves the possibility of adding meromorphic functions. However, the latter have pole singularities, which would describe sources with more singular energy-momentum tensors (of, say, $\delta'(z)$ type) which are not considered here.

Finally, one can check that time-dependent constants cannot be added to $\phi^{(1)}$ (because of the $K = 0$ condition) nor to $\beta^{(1)}$ (because of the asymptotic condition) and, as far as $\alpha^{(1)}$ is concerned, they can be readsorbed in a redefinition of the time variable. Therefore, we shall take as our starting point the solution (2.5), in which the logs are written in units of an arbitrary scale.

To solve for the motion we have to impose the three geodetic equations (1.20). First note that the time components can be integrated immediately using the expression (1.11) of $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, and yields Eq. (2.1), showing that dt/ds_i is given in terms of velocities and fields.

On the other hand, the space components of the geodetic equation have a simplified structure in this gauge because the metric at time t only depends on position and velocities and not on higher time derivatives. As a consequence, since the affine connection contains only first derivatives of the metric, Eqs.(1.20) involve at most the particles' acceleration and are thus of Newtonian type. Instead in a covariant gauge, due to the retarded propagation, all time derivatives of $\xi_i(t)$ would contribute.

In detail, in order to compute the connection components it is useful to observe that, in our parametrization, they can be cast in a simple form, which isolates some first-order contribution. First the $\Gamma^\mu_{\nu\rho}$ are expressed in terms of the $\Gamma^0_{\nu\rho}$ as follows

$$\begin{aligned}
\Gamma^z_{zz} &= \beta\Gamma^0_{zz} + 2\partial_z(\phi) \quad , \quad \Gamma^z_{\bar{z}\bar{z}} = \beta\Gamma^0_{\bar{z}\bar{z}} \ , \\
\Gamma^z_{0z} &= \beta\Gamma^0_{0z} - e^{-2\phi}\partial_z(e^{2\phi}\beta) \quad , \quad \Gamma^z_{0\bar{z}} = \beta\Gamma^0_{0\bar{z}} \ , \\
\Gamma^z_{00} &= \beta\Gamma^0_{00} - \partial_0\beta + e^{-2\phi}(\partial_{\bar{z}}\alpha^2 + \beta\partial_z(e^{2\phi}\beta) - e^{2\phi}\bar{\beta}\partial_{\bar{z}}\beta).
\end{aligned} \tag{2.6}$$

Furthermore, the $\Gamma^0_{\nu\rho}$ themselves are

$$\begin{aligned}
\Gamma^0_{zz} &= \frac{1}{2}\alpha^{-2}e^{2\phi}\partial_z\bar{\beta} \ , \\
\Gamma^0_{0z} &= \alpha^{-1}\partial_z\alpha - \beta\Gamma^0_{zz} \ , \\
\Gamma^0_{00} &= \alpha^{-1}\partial_0\alpha - \beta\Gamma^0_{0z} - \bar{\beta}\Gamma^0_{0\bar{z}}.
\end{aligned} \tag{2.7}$$

If we limit ourself to the first-order in G, all the contributions proportional to β can be neglected, and we arrive at the following equation:

$$\frac{d}{dt}\left(\frac{d\xi_1}{ds_1}\right) = \frac{d}{dt}\left(\frac{dt}{ds_1}v_1\right) = 4Gm_2\gamma_1\gamma_2\frac{(v_1-v_2)^2}{\xi_1-\xi_2}. \tag{2.8}$$

To first order accuracy, one can set in the r.h.s. of (2.8)

$$\dot{\xi} \equiv \dot{\xi}_1 - \dot{\xi}_2 = v_1 - v_2 = \frac{P\mathcal{M}}{E_1E_2} = V_0 = \text{const.}, \quad E_i = m_i\gamma_i \tag{2.9}$$

where $\mathcal{M} = E_1 + E_2$. We then obtain

$$\frac{d}{dt}\left(m_1\frac{dt}{ds_1}v_1\right) = gP\frac{\dot{\xi}}{\xi}, \qquad (g \equiv 4G\mathcal{M}) \tag{2.10}$$

and the constant of motion

$$P = P_1 = m_1\frac{dt}{ds_1}v_1(1 - g\log\xi). \tag{2.11}$$

By introducing in Eq. (2.11) the undisturbed trajectory for given impact parameter b

$$\xi^{(0)}(t) = ib + V_0t \tag{2.12}$$

we can read off the rotation angle of v_1 when t varies from $-\infty$ to $+\infty$, i.e., the first order scattering angle

$$\theta = \mp\pi g = \mp 4\pi G\mathcal{M}, \qquad (b{\gtrless}0), \tag{2.13}$$

a well-known result at this order [2], [5], [9].

By replacing in Eq.(2.1) the first order fields, we also obtain

$$\left(\frac{dt}{ds_1}\right)^2[1 - |v_1|^2 + 4Gm_2\gamma_2|v_1-v_2|^2\log|\xi|^2] = 1 \tag{2.14}$$

and thus by Eq.(2.11), a constant of motion of energy type

$$E_1 = m_1\frac{dt}{ds_1}(1 - g\frac{\bar{v}_1v_2}{2}\log|\xi|^2) \tag{2.15}$$

such that $E_1^2 - |P_1|^2 = m_1^2$. Therefore, E_1 and P_1 have the meaning of Minkowskian energy and momentum [9].

Finally, by using again Eq.(2.15) to eliminate dt/ds_1 in Eq.(2.11) we obtain a simple expression for the relative speed

$$\dot{\xi}(t) = V_0(1 + g \log \xi - g \frac{\bar{v}_1 v_2}{2} \log |\xi|^2) \tag{2.16}$$

from which the detailed first order trajectory is easily found.

3 Second order metric for any speed

At higher orders in G, the advantages of working with an instantaneous gauge show up clearly. We have already remarked in general that fields and trajectories are determined by separate equations and that the equations of motion are "Newtonian" in the sense that they are 2nd order in time at all orders. This allows to find, at n-th order, the source terms for the (n+1)-th order in a straightforward way.

In practice, at second order in G_N we need first to know the first order correction to proper time. By expanding (2.1) or (2.14) at first non trivial order we obtain, say for $i = 1$,

$$\begin{aligned} \frac{dt}{ds_1} &= \gamma_1 \left[1 + \gamma_1^2 \gamma_2 |v_1 - v_2|^2 \phi_2|_1 + ...\right] \\ &= \gamma_1 + \frac{dt^{(1)}}{ds_1} + ... \end{aligned} \tag{3.1}$$

By replacing (3.1) and (2.5) in the r.h.s. of Eqs. (1.17) we obtain the field equations in their second order form:

$$\begin{aligned} \nabla^2 \phi^{(2)} = &- \quad \partial_{\bar{z}} \beta^{(1)} \partial_z \bar{\beta}^{(1)} + \left(\alpha^{(1)} \gamma_1 + \left(\frac{dt}{ds_1}\right)^{(1)}\right) \nabla^2 \phi_1 + \\ &+ \quad \left(\alpha^{(1)} \gamma_2 + \left(\frac{dt}{ds_2}\right)^{(1)}\right) \nabla^2 \phi_2, \\ \nabla^2 \beta^{(2)} = &+ \quad 4 \partial_z (\alpha^{(1)} - 2\phi^{(1)}) \partial_{\bar{z}} \beta^{(1)} + \alpha^{(1)} \nabla^2 \beta^{(1)} + \\ &+ \quad 2 \left(\bar{v}_1 \left(\frac{dt}{ds_1}\right)^{(1)} - \gamma_1 \beta^{(1)}\right) \nabla^2 \phi_1 + 2 \left(\bar{v}_2 \left(\frac{dt}{ds_2}\right)^{(1)} - \gamma_2 \beta^{(1)}\right) \nabla^2 \phi_2, \\ \nabla^2 \alpha^{(2)} = &+ \quad 2 \partial_z \bar{\beta}^{(1)} \partial_{\bar{z}} \beta^{(1)} - \left(\gamma_1 v_1 \bar{v}_1 \phi^{(1)} + \left(\frac{dt}{ds_1}\right)^{(1)} v_1 \bar{v}_1 - \gamma_1 (v_1 \bar{\beta} + \bar{v}_1 \beta)\right) \nabla^2 \phi_1 - \end{aligned}$$

$$
- \quad \left(\gamma_2 v_2\bar{v}_2\phi^{(1)} + \left(\frac{dt}{ds_2}\right)^{(1)} v_2\bar{v}_2 - \gamma_2(v_2\bar{\beta} + \bar{v}_2\beta)\right)\nabla^2\phi_2, \tag{3.2}
$$

where we have rewritten the δ-functions in terms of Laplacians, and the first terms in the r.h.s. represent the remaining non-linear parts.

If we carefully read these expressions, using the previous solution for the fields, it appears that the sources are ill-defined, due to the presence of self-interactions, like $\phi_1\nabla^2\phi_1$. However, integrating by parts the non linear term produces similar ill-defined terms which exactly cancel those coming from the sources. At this point the inversion of the Laplacians is straightforward, even if cumbersome. After some algebra we get the following results

$$
\begin{aligned}
\phi^{(2)} = & - \gamma_1^2 v_1\bar{v}_1\frac{\phi_1^2}{2} - \gamma_2^2 v_2\bar{v}_2\frac{\phi_2^2}{2} - \gamma_1\gamma_2(v_1\bar{v}_2 + \bar{v}_1 v_2)\frac{\phi_1\phi_2}{2} + \\
& + \gamma_1\gamma_2(v_1\bar{v}_2 - \bar{v}_1 v_2)\frac{\phi_{12}}{2} + \\
& + \gamma_1\gamma_2\left[\frac{v_1\bar{v}_2 + \bar{v}_1 v_2}{2} - v_1\bar{v}_1 + \gamma_2^2(v_1 - v_2)(\bar{v}_1 - \bar{v}_2)\right]\phi_1|_2\phi_2 + \\
& + \gamma_1\gamma_2\left[\frac{v_1\bar{v}_2 + \bar{v}_1 v_2}{2} - v_2\bar{v}_2 + \gamma_1^2(v_1 - v_2)(\bar{v}_1 - \bar{v}_2)\right]\phi_2|_1\phi_1 \ ,
\end{aligned}
$$

$$
\begin{aligned}
\beta^{(2)} = & - \gamma_1^2 v_1(2 + v_1\bar{v}_1)\phi_1^2 - \gamma_2^2 v_2(2 + v_2\bar{v}_2)\phi_2^2 + \\
& - \gamma_1\gamma_2\left[2(v_1 + v_2) + v_1 v_2(\bar{v}_1 + \bar{v}_2)\right]\phi_1\phi_2 + \gamma_1\gamma_2\left[2(v_1 - v_2) - v_1 v_2(\bar{v}_1 - \bar{v}_2)\right]\phi_{12} + \\
& + \gamma_1\gamma_2\left[2(v_2 - v_1) + v_1 v_2(\bar{v}_2 - \bar{v}_1) + 2\gamma_2^2 v_2(v_1 - v_2)(\bar{v}_1 - \bar{v}_2)\right]\phi_1|_2\phi_2 + \\
& + \gamma_1\gamma_2\left[2(v_1 - v_2) + v_1 v_2(\bar{v}_1 - \bar{v}_2) + 2\gamma_1^2 v_1(v_1 - v_2)(\bar{v}_1 - \bar{v}_2)\right]\phi_2|_1\phi_1 \ ,
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{(2)} = & \quad \gamma_1^2 v_1\bar{v}_1\phi_1^2 + \gamma_2^2 v_2\bar{v}_2\phi_2^2 + \gamma_1\gamma_2(v_1\bar{v}_2 + \bar{v}_1 v_2)\phi_1\phi_2 + \\
& + \gamma_1\gamma_2(\bar{v}_1 v_2 - v_1\bar{v}_2)\phi_{12} + \\
& + \gamma_1\gamma_2\left[v_1\bar{v}_2 + \bar{v}_1 v_2 - 2v_1\bar{v}_1 - 2\gamma_1^2 v_1\bar{v}_1(v_1 - v_2)(\bar{v}_1 - \bar{v}_2)\right]\phi_2|_1\phi_1 + \\
& + \gamma_1\gamma_2\left[v_1\bar{v}_2 + \bar{v}_1 v_2 - 2v_2\bar{v}_2 - 2\gamma_2^2 v_2\bar{v}_2(v_1 - v_2)(\bar{v}_1 - \bar{v}_2)\right]\phi_1|_2\phi_2.
\end{aligned} \tag{3.3}
$$

Here the left over unknown is ϕ_{12}, which comes from inverting the Laplacian over the terms with antisymmetric product of derivatives, i.e. is defined by

$$
\nabla^2\phi_{12} \ = \ 4(\partial_z\phi_1\partial_{\bar{z}}\phi_2 - \partial_{\bar{z}}\phi_1\partial_z\phi_2). \tag{3.4}
$$

This equation can be integrated directly, but it is more convenient to impose the gauge condition $K = 0$ on the fields (3.3) from which we get, after some algebra, the constraint

$$\partial_z \phi_{12} \quad = \quad J_z = (\phi_2 - \phi_2|_1)\partial_z \phi_1 - (\phi_1 - \phi_1|_2)\partial_z \phi_2. \tag{3.5}$$

It is easy to see that (3.5) yields the Laplacian in Eq. (3.4), as expected. Furthermore, Eq. (3.5) can be used to construct the solution

$$\phi_{12}(z, \bar{z}) = \int^{(z,\bar{z})} \ (dz J_{\overline{z}} \ - \ d\overline{z} J_z), \quad \mathbf{J} = (J_z, J_{\bar{z}}), \tag{3.6}$$

which is automatically single-valued because $\mathbf{J}$ is divergenceless, as a consequence of the subtraction of $\phi_1|_2$ and $\phi_2|_1$ in Eq. (3.5).

The explicit solution for ϕ_{12} (Appendix B) can be written as function of the complex variable

$$Z \equiv \frac{z - \xi_1}{\xi_2 - \xi_1}, \tag{3.7}$$

with $z = x + iy$ and $\xi_i = \xi_i^x + i\xi_i^y$, in the form

$$\begin{aligned} \phi_{12}(\mathbf{x}, \xi_1, \xi_2) = \quad & 4G^2 m_1 m_2 \Big(- \log(1 - Z) \log \bar{Z} + \log(1 - \bar{Z}) \log Z + \\ + \quad & Li_2(1 - Z) + Li_2(\bar{Z}) - Li_2(Z) - Li_2(1 - \bar{Z}) \Big), \end{aligned} \tag{3.8}$$

where $Li_2(z)$ denotes the Spencer's function [14]. Using the above expression we can compute the time derivative of ϕ_{12} which can be written in terms of ϕ_1 and ϕ_2:

$$\begin{aligned} \partial_0 \phi_{12} = \quad & \phi_1 (v_2 \partial_z - \bar{v}_2 \partial_{\bar{z}}) \phi_2 - \phi_2 (v_1 \partial_z - \bar{v}_1 \partial_{\bar{z}}) \phi_1 + \\ - \quad & \phi_1|_2 (v_2 \partial_z - \bar{v}_2 \partial_{\bar{z}}) \phi_2 + \phi_2|_1 (v_1 \partial_z - \bar{v}_1 \partial_{\bar{z}}) \phi_1 + \\ + \quad & \left((v_1 - v_2)(\partial_z \phi_2)|_1 - (\bar{v}_1 - \bar{v}_2)(\partial_{\bar{z}} \phi_2)|_1 \right) \phi_1 + \\ + \quad & \left((v_1 - v_2)(\partial_z \phi_1)|_2 - (\bar{v}_1 - \bar{v}_2)(\partial_{\bar{z}} \phi_1)|_2 \right) \phi_2. \end{aligned} \tag{3.9}$$

These relations are useful to check that the gauge condition (1.15) is satisfied, once the first order geodetic motion is taken into account.

4 Equations of motion and scattering angle

Studying the equations of motion (1.20) involves replacing the cumbersome second order fields (3.3) in the expressions (2.6) and (2.7) for the affine connection, and gives rise to a rather lengthy algebra. The latter is however simplified by the following observations:

1) All singular terms containing at least one field ϕ_i, evaluated at the source, should cancel out. In other words, there are no self-interactions.

2) In the r.h.s. of (1.20) one can use the first order equations of motion, which involve several conserved quantities, described in Eqs. (2.11) and (2.15). In particular one can define a c.m. frame in which

$$m_1 \frac{dt}{ds_1} v_1 + m_2 \frac{dt}{ds_2} v_2 = 0, \tag{4.1}$$

at least up to second order in Gm_i.

More precisely, after doing the algebra mentioned before, we arrive at the following equation for , say, the spatial components of particle 1

$$\begin{aligned} \frac{d}{ds_1}\left(\frac{dt}{ds_1} v_1\right) = & \quad (\frac{dt}{ds_1})^2 \Big[2\gamma_2 (v_1 - v_2)^2 (\partial_z \phi_2)|_1 (1 + \gamma_1 \gamma_2^2 |v_1 - v_2|^2 \phi_1|_2 - \gamma_1 v_1 (\bar{v}_1 - \bar{v}_2)\phi_1|_2) + \\ & + \quad 2\gamma_2 v_2 (\partial_{\bar{z}} \phi_2)|_1 (\bar{v}_1 - \bar{v}_2)^2 (\gamma_2 v_2 \phi_2|_1 + \gamma_1 v_1 \phi_1|_2) \Big], \end{aligned} \tag{4.2}$$

while the time component can be replaced by the expressions of dt/ds_1 and dt/ds_2 obtained from Eq. (3.1).

To second order accuracy one can use the first order equations of motion in the r.h.s. of (4.2), and in particular the expression (3.1) for dt/ds_2 and the center of mass frame condition (4.1), to obtain

$$\frac{d}{dt}\left(m_1 \frac{dt}{ds_1} v_1\right) = 4Gm_1 m_2 \frac{dt}{ds_1}\frac{dt}{ds_2}\frac{(v_1 - v_2)^2}{\xi_1 - \xi_2}(1 - \gamma_1 v_1 (\bar{v}_1 - \bar{v}_2)\phi_1|_2). \tag{4.3}$$

The discussion of this equation can be further simplified by introducing the Minkowskian energies E_1 and E_2 and momentum P, which appear as first-order constants of motion in Eqs. (2.11) and (2.15). By using the notation

$$V_1 = \frac{P}{E_1}, \quad V_2 = -\frac{P}{E_2}, \quad \mathcal{M} = E_1 + E_2, \quad g = 4G\mathcal{M}, \ \xi = \xi_1 - \xi_2 \tag{4.4}$$

Eq. (4.3) can be rewritten as

$$\frac{d}{dt}\left(m_1\frac{dt}{ds_1}v_1\right) = 4GE_1E_2\frac{\dot{\xi}^2}{\xi}\left(1+\frac{g}{2}\bar{V}_1V_2\log|\xi|^2\right), \tag{4.5}$$

where, by the first order equation (2.14), we can set

$$\dot{\xi} = (V_1-V_2)(1+g\log\xi-\frac{g}{2}\bar{V}_1V_2\log|\xi|^2)+O(G^2). \tag{4.6}$$

By finally replacing (4.6) in the r.h.s. of (4.5) we obtain

$$\frac{d}{dt}\left(m_1\frac{dt}{ds_1}v_1\right) = gP\frac{\dot{\xi}}{\xi}(1+g\log\xi), \tag{4.7}$$

which can be integrated to yield

$$p_1(t) \equiv m_1\frac{dt}{ds_1}v_1 = P(1+g\log\xi+\frac{1}{2}g^2(\log\xi)^2). \tag{4.8}$$

In conclusion, the "momentum" variable $p_1(t)$, as function of the relative distance ξ, just exponentiates the first order result and, up to second order included, has by (4.6) and (4.8) the form

$$p_1(t) = P\xi^g \simeq P(V_0t+ib)^{g(1+g)}|V_0t|^{-g^2\bar{v}_1v_2}, \qquad (|V_0t|\gg b). \tag{4.9}$$

From the large time behaviour in (4.9) we can read off the second order scattering angle

$$\theta = \mp\pi g(1+g)+O(g^3), \qquad g\equiv 4G\mathcal{M},\ (b>0)\ \ ((b<0)), \tag{4.10}$$

an expression which can be checked by explicit integration of Eq. (4.9) to yield $\xi_1(t)$ at all times.

The results (4.9) and (4.10) call for several comments. Firstly, the impressive simplification of nonlinearities in this gauge is presumably rooted in a simple relation to the Minkowskian (singular) gauge [2], [9] which may hold in this case. In fact the present instantaneous gauge is actually equivalent to a Coulomb-type gauge [6], [15] in a first order (Palatini) formalism and this may provide a basis for a non-perturbative construction [16] of dreibein and metric to all orders. Secondly, the expression (4.10) shows no explicit m_i dependence at fixed total invariant mass $\mathcal{M}$, which in turn coincides with the topological invariant [5] at this order. Thus, there is a smooth massless limit and there are second order

corrections to the scattering angle even in the massless case. This is in agreement with suggestions by 't Hooft [6] , and is at variance with previous findings by Cappelli, and two of us [9] in covariant-type gauges, which provide an alternative definition of c.m. frame, [17].

Although disappointing, the gauge dependence of the scattering angle noticed above is not terribly surprising because the instantaneous gauge changes in a profound way the relation of two-body vs. one-body metrics: in particular there is no simple way of decoupling particles at large space separations due to the presence of logaritmically increasing fields. This is to be contrasted to what happens in covariant-type gauges [9], [17] where such decoupling is built in and gives rise, in the massless limit, to scattering of Aichelburg-Sexl type.

The above remarks show that further study of our conformal type gauge is needed, possibly at non perturbative level, in order to better investigate the role of asymptotic conditions in the scattering problem.

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A - Constraints vs. equations of motion

In order to discuss the consistency of the instantaneous gauge with the geodetic motion, it is useful to recast the field equations (1.17) in terms of new variables $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h^{\alpha}_{\alpha} \eta_{\mu\nu}$, where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ with the property $\gamma_{12} = 0, \gamma_{11} = \gamma_{22}$. These can be rewritten as four basic equations

$$\begin{aligned} \frac{1}{2}\nabla^2(\gamma_{00} - \gamma_{11}) &= \Lambda_{00} + gT_{00} = \tilde{T}_{00} \\ \frac{1}{2}\nabla^2\gamma_{0i} &= \Lambda_{0i} + gT_{0i} = \tilde{T}_{0i} \\ \nabla^2\gamma_{11} &= \Lambda_{xx} + \Lambda_{yy} + g(T_{xx} + T_{yy}) = (\tilde{T}_{xx} + \tilde{T}_{yy}) \end{aligned} \tag{A.1}$$

where the tensor $\Lambda_{\mu\nu}$ is given by

$$\Lambda_{\mu\nu} = \frac{1}{4} \eta_{\mu\mu'} \eta_{\nu\nu'} \epsilon^{\mu'\rho\sigma} \epsilon^{\nu'\gamma\delta} g_{\alpha\beta} \left[\Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\beta}_{\rho\delta} - \Gamma^{\alpha}_{\sigma\delta}\Gamma^{\beta}_{\rho\gamma} \right], \tag{A.2}$$

and the modified energy-momentum tensor $\tilde{T}_{\mu\nu}$ satisfies the trivial conservation law $\partial_\mu \tilde{T}^{\mu\nu} = 0$ equivalent to the covariant conservation of $T^{\mu\nu}$.

The other two Einstein equations give constraints on the integration of the four variables $\gamma_{\mu\nu}$. These can be summarized in one complex equation

$$G_{zz} = \partial_z(\partial_0\gamma_{0z} - \partial_z\gamma_{11}) - \Lambda_{zz} - gT_{zz} = 0. \tag{A.3}$$

The gauge condition $K = 0$ can also be rewritten as

$$\partial_i \gamma_{0i} = \partial_0(\gamma_{00} - \gamma_{11}) \tag{A.4}$$

A consistency test is provided by imposing that the Laplacian of the gauge condition (A.4) and of the complex equation (A.3) is zero. Using the first four equations of motion (A.1) we get

$$\frac{1}{2}\nabla^2[\partial_0(\gamma_{00} - \gamma_{11}) - \partial_i(\gamma_{0i})] = \partial_\alpha \tilde{T}_{0\alpha} = 0 \tag{A.5}$$

$$\nabla^2(G_{zz}) = \partial_z \partial_\alpha \tilde{T}_{z\alpha} = 0 \tag{A.6}$$

Hence (A.5) and (A.6) show that, for every solution of the first four equations, imposing the covariant conservation of $T^{\mu\nu}$, equivalent to the geodesic equations (1.20)implies that the gauge condition and the constraint on G_{zz} are simply the sum of pure analytic and anti-analytic functions. Requiring that the connections vanish at spatial infinity, i.e. imposing that $K = 0$ and $G_{zz} = 0$ as a boundary condition, is then enough to ensure that these equations are satisfied in the whole two-dimensional plane, as stated in Sec. II.

B - Monodromic solution for Poisson-like equation

In the following we show how to construct a single-valued solution to the Poisson-like equation (3.4)in two spatial dimensions avoiding the nasty calculations implied by the more general Green-function method.

Since $\nabla^2 = 4\partial_z\partial_{\bar{z}}$, it is easy to obtain a particular solution of such type of equations by just integrating in z and $\bar{z}$ the source, which is given as a sum of terms with factorized dependence on z and $\bar{z}$.

The integration may produce unwanted polydromy. Since the source is monodromic, then the polydromic terms have simple discontinuities which are analytic or anti-analytic functions and can be eliminated by exploiting the arbitrariness in the inversion of the Laplacian [18].

In our case, by integrating eq. (3.4) we obtain the following particular solution

$$(4G^2 m_1 m_2)^{-1}\ \phi^0_{12} = \log(z - \xi_1)\log(\bar{z} - \bar{\xi}_2) - (1 \leftrightarrow 2) \tag{B.1}$$

If we circle particle 1, then the r.h.s. of (B.1) gets an additional contribution

$$+\ 2\pi i[\log(\bar{z} - \bar{\xi}_2) - \log(z - \xi_2)]. \tag{B.2}$$

To compensate for the previous contribution we need to add the following harmonic function which has the opposite discontinuity around particle 1 of the particular solution and no discontinuity around particle 2:

$$h_1 = Li_2(1-Z) - \log(Z)\log(\bar{\xi}_1 - \bar{\xi}_2) - c.c., \tag{B.3}$$

where the Spencer function $Li_2(z)$ has a branch-point at $z = 1$ and is defined by

$$Li_2(z) = -\int_0^z \frac{dx}{x}\ln(1-x) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}, \tag{B.4}$$

and we remind that $Z = (z-\xi_1)/(\xi_2-\xi_1)$. Similarly to compensate for the discontinuity of ϕ^0_{12} around the particle 2, we need to add an other harmonic function

$$h_2 = -Li_2(Z) + \log(1-Z)\log(\bar{\xi}_2 - \bar{\xi}_1) - c.c.. \tag{B.5}$$

By adding (B.3) and (B.5) to the r.h.s. of (B.1), we get the complete single-valued solution ϕ_{12} given in Eq. (3.8) of the text.

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